Vol. 4, No. 3, 2024, 1-9. DOI: 10.47909/ijsmc.136

ORIGINAL ARTICLE

Majorization and the degree sequence of trees

Leo Egghe[1], **Ronald Rousseau**[2]

[1] University of Hasselt, Belgium.
[2] Faculty of Social Sciences, University of Antwerp, Belgium.
Department of MSI, Centre for R&D Monitoring (ECOOM), Belgium.
Email: ronald.rousseau@uantwerpen.be. **ORCID:** https://orcid.org/0000-0002-3252-2538.
Corresponding author.

ABSTRACT

Objective. This study investigated the relation between the degree sequences of trees and the majorization order.

Design/Methodology/Approach. The majorization technique was employed in accordance with the tenets of Muirhead's theorem.

Results. In this study, we proved a theorem that provides a necessary and sufficient condition for degree sequences of trees to be comparable in the majorization order.

Research Limitations. Our research was focused on the study of trees rather than general networks. Furthermore, our investigation was primarily theoretical in nature.

Practical Implications. Given the pervasiveness of trees in the field of information science, our theoretical study made a significant contribution to the advancement of knowledge regarding trees as a crucial data structure.

Originality/Value. This study represented a rare instance of a text that combines two distinct but related areas of study: the Lorenz curves and majorization on the one hand and the degree sequences of networks on the other.

Keywords: networks; trees; data structures; majorization; Lorenz curves; degree sequences.

1. INTRODUCTION

IN THIS introduction, we recall the fundamental concepts that will be discussed in greater detail in the subsequent sections of this article. These notions and their associated notation are well-established in the fields of network and graph theory (see, e.g., Knuth, 1973; Wasserman & Faust, 1994) or are taken from previous articles (Egghe, 2024). Let $G = (V,E)$ be an undirected network, where $V = (vk)_{k=1,\dots,N}$ denotes the set of nodes or vertices, and E denotes the set of links or edges. We assume that $\#V = N > 1$. A path of length n is a sequence of vertices $(v_0, \dots, v_k, v_{k+1}, \dots, v_n)$ such that $\{v_0, \dots, v_{n-1}\}$ and $\{v_1, \dots, v_n\}$ are sets (being sets each consist of different elements), and for $k = 0, \dots, n-1$, v_k is adjacent to v_{k+1}. A cycle is a path for which the starting point v_0 coincides with the endpoint v_n. A graph is connected if there exists (at least one) path between any two vertices. If $\#V = N$, then the degree of node i ($i = 1, \dots, N$; i.e., the

Received: 01-09-2024. **Accepted:** 22-10-2024. **Published:** 06-11-2024.

How to cite: Egghe, L., & Rousseau, R. (2024). Majorization and the degree sequence of trees. *Iberoamerican Journal of Science Measurement and Communication; 4*(3), 1-9. DOI: 10.47909/ijsmc.136

number of edges connected to node i) is denoted as δ_i. In this article, we always assume that G is connected; hence, all degrees are strictly larger than zero. As there is no natural order among the nodes in a network, we assume that these values are ranked in decreasing order.

1.1. Notation

The sequence of degrees of the nodes in a network G with N nodes is denoted as

$$\Delta_G = (\delta_1(G), \delta_2(G), \ldots, \delta_N(G)), \quad (1)$$

We will informally refer to such a sequence as a delta sequence, consisting of delta values. Indices in the delta notation refer to a rank. Clearly, $\sum_{i=1}^{N} \delta_i = 2(\#E)$, a notion which is known as the total degree of the network. It is easy to see that $2(N-1) \le \sum_{i=1}^{N} \delta_i \le N(N-1)$. The lower bound is obtained, for example, for a tree (hence also for a chain) consisting of N nodes, while the upper bound is obtained for a complete graph where each node is connected to all other nodes. Before moving on to examples and theories, we recall the following definitions.

1.2. Definition: Trees and branches

A free or unrooted tree is a connected graph with no cycles. Equivalently, it is a connected graph such that removing any edge makes it disconnected. Another equivalent definition states that if v and v' are different vertices, then there exists exactly one path from v to v' (Knuth, 1973). Often, there is one designated node, called “the root.” In that case, one says that the tree is rooted.

If m is any node in T (but not a terminal node, i.e., a node with degree one), then a branch rooted at m consists of one link at m, and all nodes and links connected to m in T via that link. This is illustrated in Figure 1.

As emphasized by Knuth (1973, p. 305), trees are the most important nonlinear data structures.

1.3. Definition: Isomorphic graphs

Two graphs G and G' are isomorphic if there exists a bijection f between the vertices of G and G' such that there is an edge between vertices u and v in G if and only if there is an edge between the vertices $f(u)$ and $f(v)$ in G'. When talking about a network or a tree, we always mean the equivalence class of isomorphic networks or trees. Hence, we do not distinguish between isomorphic networks. By definition, two isomorphic networks have the same delta sequence, but the opposite is not true (Egghe, 2024).

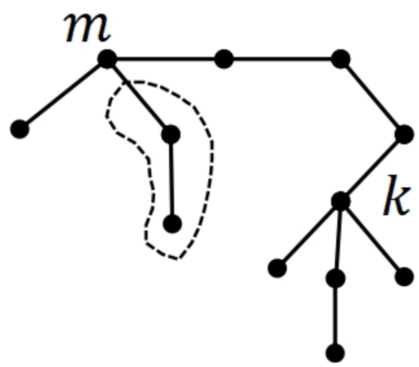

Figure 1. A tree and a branch rooted in node *m*.

1.4. Definition: Spanning tree of a connected graph

A spanning tree of an N-node connected graph is a set of $N-1$ edges that connects all nodes of the network and contains no cycles. A graph may have different (non-isomorphic) spanning trees.

1.5. Definition: The Lorenz curve

Let $X = (x_1, x_2, \ldots, x_N)$ be an N-sequence with $x_j \in \mathbb{R}^+$, $j = 1, \ldots, N$ (Lorenz, 1905). If X is an N-sequence, ranked in decreasing order (always used in the sense that ranking is not necessarily strict), then the Lorenz curve of X is the curve in the plane obtained by the line segments connecting the origin (0,0) to the points $\left(\frac{k}{N}, \frac{\sum_{j=1}^{k} x_j}{\sum_{j=1}^{N} x_j}\right)$, $k = 1, \ldots, N$. For $k = N$, the endpoint (1,1) is reached.

1.6. Definition: The majorization property

If X and X' are N-sequences, ranked in decreasing order, then X is majorized by X' (equivalently, X' majorizes X), denoted as $X \preccurlyeq_L X'$, if (Hardy *et al.,* 1934; Marshall *et al.,* 2011)

$$\sum_{j=1}^{N} x_j \le x_j' \text{ for } k = 1, \ldots, N-1 \text{ and } \sum_{j=1}^{N} x_j = \sum_{j=1}^{N} x_j' \quad (2)$$

The index L in $X \preccurlyeq_L X'$ refers to the fact that this order relation corresponds to the order relation between the corresponding Lorenz curves. One may observe that X is majorized by X' ($X \preccurlyeq_L X'$) if and only if the Lorenz curve of X' is situated above (or coincides with) the Lorenz curve of X. It is well-known (see, e.g., Marshall *et al.,* 2011, p. 14) that $X \preccurlyeq_L Y$ is equivalent to each of the following statements:

(A) $\sum_i \varphi(x_i) \leq \sum_i \varphi(y_i)$ for all continuous, convex functions $\varphi: \mathbb{R} \to \mathbb{R}$.
(B) Y can be obtained from X by a finite number of elementary transfers (Muirhead, 1903).

Here, an elementary transfer is a transformation from $(x_1, \ldots, x_N)$, where $(x_1, \ldots, x_N)$ is ranked in decreasing order, into $(x_1, \ldots, x_i + h, \ldots, x_j - h, \ldots, x_N)$, where $0 < h \leq x_j$.

1.7. Definition: Basic transfers

In the case that the elements in $(x_1, \ldots, x_N)$ are natural numbers, h can also be taken as a natural number, and it can even be taken to be equal to 1. In this case, we will say that this transfer is a basic transfer. The appendix shows how to perform such basic transfers. We write $X \prec_L Y$ for the strict Lorenz majorization, that is, $X \preccurlyeq_L Y$ with $X \neq Y$.

1.8. Definition: Non-normalized Lorenz curves

Let $X = (x_1, x_2, \ldots, x_N)$ be a decreasing N-sequence of nonnegative real numbers, then the corresponding non-normalized Lorenz curve is the polygonal line connecting the origin (0,0) with the points $(j, \sum_{k=1}^{j} x_j)$, $j = 1, \ldots, N$. This curve ends at the point with coordinates $(N, \sum_{k=1}^{j} x_j)$.

1.9. Definition: The non-normalized (or generalized) majorization order for *N*-sequences

If X and Y are decreasing N-sequences of nonnegative real numbers, then X is majorized by Y, denoted as $X \preccurlyeq Y$ if

$$\forall j, j = 1, \ldots, N: \sum_{k=1}^{j} x_k \leq \sum_{k=1}^{j} y_k \quad (3)$$

The relation $\preccurlyeq$ is only a partial order as non-normalized Lorenz curves (just like standard Lorenz curves) may intersect. If $x_j \leq y_j$, for $j = 1, \ldots, N$, then obviously $X \preccurlyeq Y$, but the opposite relation does not hold. As for the Lorenz majorization, we write $X \prec Y$, for $X \preccurlyeq Y$ with $X \neq Y$.

2. TREES IN THE INFORMATION SCIENCES

Without aiming to be exhaustive, we present a few examples to demonstrate that trees have frequently been utilized in the information sciences. Every hierarchy leads to a tree structure. For instance, hierarchical clustering algorithms lead to special rooted trees called "dendrograms." Each terminal node of a dendrogram represents an object; nonterminal nodes represent non-singleton clusters; and the root represents the entire object set. Similarly, every ontology or classification scheme such as the MeSH index (Leydesdorff *et al.,* 2012) and those discussed or used by Kwasnik (1999) and Archambault *et al.,* (2011) is a tree, while trees have also been used for name disambiguation (Wang *et al.,* 2012). Further, spanning trees have frequently been applied (Yang *et al.,* 2012).

A well-known tree in citation analysis is the Chubin–Moitra citation classification scheme (Chubin & Moitra, 1975; Rousseau *et al.,* 2018, p. 109). Here, citations (the root) are first subdivided into the affirmative and the negative branches. Terminal nodes on the affirmative branch are basic, subsidiary, additional, and perfunctory citations. On the negative branch, there are partially and totally negative citations. Another type of tree used in the information sciences is the decision tree. This is a tree structure that enables a decision maker to decompose a large complex decision problem into several smaller problems (Winston, 1994). Such trees have been used, for example, in Fidel (1991), Kenekayoro *et al.,* (2014), Ma *et al.,* (2013), and Zheng *et al.,* (2006). We finally recall that Kosmulski (2013) proposed a family tree of bibliometric indices.

3. BASIC TRANSFERS AND DELTA SEQUENCES OF TREES

First, we explain the relation between a basic transfer and the delta sequence of a tree. Given a tree T with a delta sequence $\Delta_T = (\delta_1(T), \delta_2(T),$

… , $\delta_N(T)$), we know that always $\delta_N(T) = 1$. If now we perform a basic transfer, replacing the sequence ($\delta_1(T)$, $\delta_2(T)$, …, $\delta_N(T)$) by ($\delta_1(T)$, …, $\delta_i(T) + 1$, …, $\delta_j(T) - 1$, …, $\delta_N(T)$), where $\delta_j(T) > 1$, we refer to this transfer as a basic tree transfer. Then, we see that the degree of the node at rank j decreased by 1. This happens if we remove a branch (without the root node) from the node at rank j. As the degree of the node at rank i has increased by 1 and all other degrees have stayed invariant, this can be realized by attaching a branch rooted at the node at rank j to the node at rank i. This is illustrated in Figure 2 and in more detail in the appendix.

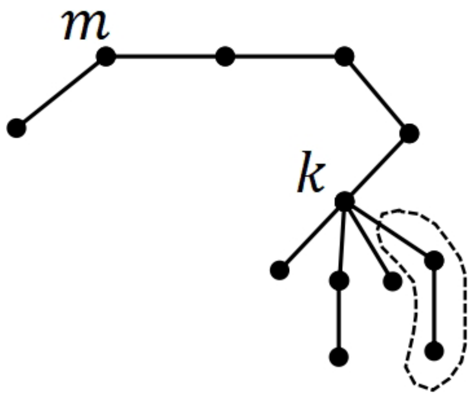

Figure 2. A branch at node *m* (see Figure 1) is replaced by the same branch placed at node *k*.

4. MAJORIZATION AND GENERALIZED MAJORIZATION BETWEEN DELTA SEQUENCES OF NETWORKS

It is well-known that the relation $\preccurlyeq$ is not a total order between delta sequences of networks (of course, with an equal number of nodes; Egghe, 2024). We next show that this is not even true for trees. Consider the following five non-isomorphic trees with $N = 8$ nodes (Figure 3).

The first four trees have the same delta sequence, namely $\Delta = (5,2,2,1,1,1,1,1)$ while the delta sequence of the last one is $\Delta' = (4,4,1,1,1,1,1,1)$. Clearly, Δ and Δ' are different and not comparable: $\Delta \not\preccurlyeq \Delta'$ and $\Delta' \not\preccurlyeq \Delta$.

Proposition 1: *In the set of delta sequences of N-node trees, the majorization order coincides with the generalized majorization* ($\preccurlyeq\ =\ \preccurlyeq_L$) *and* ($\prec\ =\ \prec_L$).

Proof: This follows immediately from the fact that the total degree of every tree with N nodes is $2(N - 1)$.

Now, we come to the main theorem of this article that provides a necessary and sufficient condition for delta sequences of trees to be comparable.

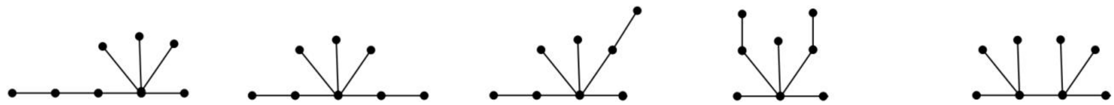

Figure 3. Five non-isomorphic trees with eight nodes.

Theorem: *Given two sequences Δ and Δ' of length N, where Δ is the delta sequence of a tree (hence, does not contain a zero) and also Δ' does not contain a zero, then*

$$\Delta \prec \Delta'$$
$$\Leftrightarrow$$

For every tree T with delta sequence Δ, there exists a tree T′ with delta sequence Δ', which is created from the tree T by moving a finite number of branches to nodes with a higher or equal degree.

Proof: Assume that the tree T' is created from the tree T by moving a finite number of branches (each without their root) to a node with a higher or equal degree. If we replace in T with $\Delta T = (\delta_1, \delta_2, \ldots, \delta_N)$ a branch of the node at rank j ($\delta_j > 1$) to the node at rank i where $\delta_i \geq \delta_j$, then only two values in ΔT change (but we have no information about the new ranking): δ_i becomes $\delta_i + 1$, and δ_j becomes $\delta_j - 1$. The new delta sequence has values

$$(\delta_1, \ldots, \delta_{i-1}, \delta_i + 1, \delta_{i+1}, \ldots, \delta_{j-1}, \delta_j - 1, \delta_{j+1}, \ldots, \delta_N) \quad (4)$$

perhaps in a different order. Anyway, the new delta sequence is strictly larger (in the $\prec\ =\ \prec_L$ ordering) than Δ_T. Performing this operation a finite number of times proves that $\Delta \prec \Delta'$.

Conversely, we consider a tree T with a delta sequence Δ (we know that such a tree exists). Hence, the given sequence Δ is ΔT. We know

now that $\Delta \prec \Delta'$. By Muirhead's theorem, we can apply a finite number of basic transfers on the tree T (moving from Δ to Δ'). The resulting tree is the tree T', whose existence we have to prove.

Figures 1 and 2 illustrate this theorem with Δ = (4,3,2,2,2,2,1,1,1,1,1) and Δ' = (5,2,2,2,2,2,1,1,1,1,1). The same reasoning as used in the theorem can be used to prove the following well-known result.

Proposition 2 (Hakimi, 1962): *Given a sequence S of length N, consisting of strictly positive numbers, and with total degree 2(N − 1), then we can construct a tree T such that S is the degree sequence of T, that is,* $S = \Delta_T$.

Proof: If C denotes the degree sequence of the N-node chain, then $C \prec L_S$, using that they both have a total degree of $2(N-1)$. Now, by Muirhead's theorem, we can apply a finite number of basic transfers to C and reach S. The resulting tree is the tree T, whose existence we have to show.

Remark 1: Proposition 2 is Corollary 1, p. 499 in Hakimi (1962).

Remark 2: The obtained tree T' does not have to be unique, as illustrated in the appendix.

Remark 3: The theorem states, "for all T with delta sequence Δ, there exists a tree T' with delta sequence Δ'." The theorem is false when this expression is replaced by "for all T and T' with delta sequences $\Delta_T = \Delta$ and $\Delta_{T'} = \Delta'$." We provide an example for $N = 8$. Let T be the tree shown in Figure 4(a) and T' the tree shown in Figure 4(b). Then $\Delta = \Delta_T$ = (4,2,2,2,1,1,1,1), $\Delta' = \Delta_{T'}$ = (5,2,2,1,1,1,1,1), and $\Delta \preccurlyeq \Delta'$. Yet, it is impossible to transform T into T' via basic transformations (recall the condition $\delta_j \geq \delta_i$).

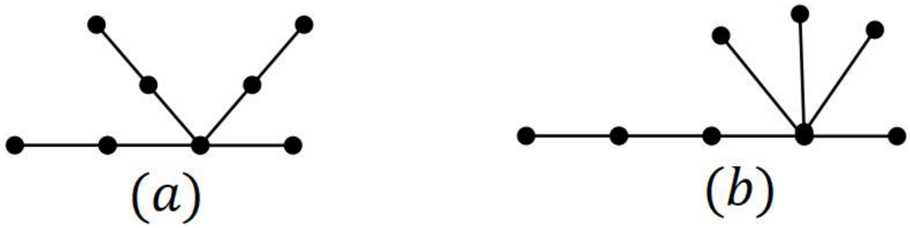

Figure 4. Trees (a) T and (b) T'.

Using this example, we can construct an illustration of the theorem though. Consider the tree T'' in Figure 5.

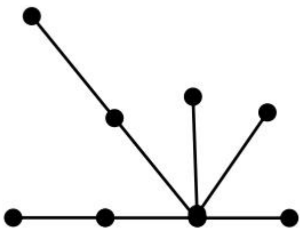

Figure 5. Tree T''.

Then $\Delta'' = \Delta_{T''}$ = (5,2,2,1,1,1,1,1) and T'' can be obtained from T by basic transfers.

Remark 4: The relation $\preccurlyeq \, = \, \preccurlyeq_L$ is a total order for trees if and only the number of nodes $N \leq 7$.

Proof: All non-isomorphic trees for $N < 23$ can be found at https://users.cecs.anu.edu.au/~bdm/data/trees.html. Then one can check that for $N \leq 7$, we have a total order. For $N = 8$, we have already given an example that the order is not total. Based on this example, it is easy to construct examples for all $N > 8$ (see Figure 6). This figure gives a tree T (left) with $\Delta_T = (5,2,\underbrace{\ldots}_{(N-6\text{times})},2,1,1,1,1,1)$ and a tree T' (right) with $\Delta_{T'} = (4,4,2,\underbrace{\ldots}_{(N-8\text{times})},2,1,1,1,1,1,1)$. Then $\Delta_T \not\preccurlyeq \Delta_{T'}$ and $\Delta_{T'} \not\preccurlyeq \Delta_T$

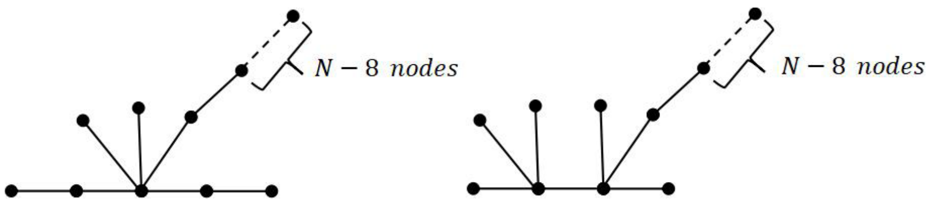

Figure 6. Non-isomorphic and non-comparable trees for $N > 8$.

Proposition 3: *If Δ_C is the delta sequence of the N-node chain and Δ_M is the delta sequence of a connected N-node network M (not being a chain), then*

$$\Delta_C \prec \Delta_M$$

Proof: Let Δ_T be the delta sequence of any tree T. Then, we know that we can obtain this tree T be a finite number of basic transfers from a chain and, hence, by the theorem $\Delta_C \preccurlyeq \Delta_T$, with equality only if T is a chain. Consider now any N-node network M, then this network has a spanning tree T_M with $T_M \preccurlyeq \Delta_M$. By the transitivity of $\preccurlyeq$ and the fact that M is not a chain, we obtain that $\Delta_C \prec \Delta_M$.

5. APPLICATIONS

In the context of data file systems, directories and files are frequently represented as a tree structure. The transfer of a subdirectory (and all its constituent files) from one directory to another is a common operation. In the terminology of this article, if the target directory has a higher degree than the original, this signifies that the new situation majorizes the old one. It is not uncommon for suggestions to be made regarding the restructuring of the fossil record, phylogenetic trees, which can result in changes that are analogous to the rearrangement of branches in a tree (see, e.g., Tschopp *et al.,* 2015).

6. CONCLUSION

This article represents a rare instance of combining Lorenz curves and majorization on the one hand and degree sequences of networks on the other. This supports Rousseau's claim (Rousseau, 2011) that Lorenz curves (and hence the majorization order) are universal tools for studying networks. In particular, we proved a theorem that provides a necessary and sufficient condition for delta sequences of trees to be comparable in the majorization order. Our methodology leads to an almost trivial proof of Hakimi's corollary (of a much more general result about linear networks) on the realizability of a set of strictly positive natural numbers as degrees of the vertices of a tree. As trees are ubiquitous in the information sciences, our study contributes to a better understanding of trees as an important data structure.

Acknowledgments

The author thanks Li Li (Beijing) for drawing excellent illustrations and Raf Guns (Antwerp) for useful discussions.

Conflict of interest

The authors declare no conflict of interest.

Contribution statement

Conceptualization, investigation, validation, writing-original draft, writing-review & editing: Leo Egghe.

Validation, writing-review & editing: Ronald Rousseau.

Statement of data consent

No data were generated in this research.

REFERENCES

Archambault, E., Beauchesne, O. H., & Caruso, J. (2011). Towards a multilingual, comprehensive and open scientific journal ontology. In B. Noyons, P. Ngulube, & J. Leta (Eds.) *Proceedings of the ISSI 2011 conference* (pp. 66-77). ISSI & University of Zululand.

Chubin, D. E. & Moitra, S. D. (1975). Content analysis of references: Adjunct or alternative to citation counting? *Social Studies of Science, 5*(4), 423-441. https://doi.org/10.1177/030631277500500403

Egghe, L. (2024). Networks and their degree distribution, leading to a new concept of small worlds. *Journal of Informetrics, 18*(3), article 101554. https://doi.org/10.1016/j.joi.2024.101554

Fidel, R. (1991). Searchers' selection of search keys: I. The selection routine. *Journal of the American Society for Information Science, 42*(7), 490-500. https://doi.org/10.1002/(SICI)1097-4571(199108)42:7

Hakimi, S. L. (1962). On realizability of a set of integers as degrees of the vertices of a linear graph I. *Journal of the Society for Industrial and Applied Mathematics, 19*(3), 496-507. https://doi:10.1137/0110037

Hardy, G. H., Littlewood, J. E., & Pólya, G. (1934). *Inequalities*. Cambridge University Press.

Kenekayoro, P., Buckley, K., & Thelwall, M. (2014). Automatic classification of academic web page types. *Scientometrics, 101*(2), 1015-1026. https://doi.org/10.1007/s11192-014-1292-9

Knuth, D. E. (1973). *The art of computer programming, Vol. 1: Fundamental algorithms* (2nd ed.). Addison-Wesley.

Kosmulski, M. (2013). Family-tree of bibliometric indices. *Journal of Informetrics, 7*(4), 313-317. https://doi:10.1016/j.joi.2012.12.002

Kwasnik, B. H. (1999). The role of classification in knowledge representation and discovery. *Library Trends, 48*(1), 22-47.

Leydesdorff, L., Rotolo, D., & Rafols, I. (2012). Bibliometric perspectives on medical innovation using the Medical Subject Headings (MeSH) of PubMed. *Journal of the American Society for Information Science and Technology, 63*(11), 2239-2253. https://doi.org/10.1002/asi.22715

Lorenz, M. O. (1905). Methods of measuring the concentration of wealth. *Publications of the American Statistical Association, 9,* 209-219. https://doi.org/10.1080/15225437.1905.10503443

Ma, Z. Y., Sun, A. X., & Cong, G. (2013). On predicting the popularity of newly emerging hashtags in Twitter. *Journal of the American Society for Information Science and Technology, 64*(7), 1399-1410. https://doi.org/10.1002/asi.22844

Marshall, A. W., Olkin, I., & Arnold, B. C. (2011). *Inequalities: Theory of majorization and its applications*. Springer.

Muirhead, R. F. (1903). Some methods applicable to identities and inequalities of symmetric algebraic functions of *n* letters. *Proceedings of the Edinburgh Mathematical Society, 21,* 144-157. https://doi.org/10.1017/S001309150003460X

Rousseau, R. (2011). Lorenz curves determine partial orders for comparing network structures. *DESIDOC Journal of Library & Information Technology, 31*(5), 340-347. https://doi.org/10.14429/djlit.31.5.1190

Rousseau, R., Egghe, L., & Guns, R. (2018). *Becoming metric-wise. A bibliometric guide for researchers*. Chandos-Elsevier.

Tschopp, E., Mateus, O., Benson, R. B. J. (2015). A specimen-level phylogenetic analysis and taxonomic revision of Diplodocidae (Dinosauria, Sauropoda). *PeerJ, 3,* Article e857. https://doi.org/10.7717/peerj.857

Wang, J., Berzins, K., Hicks, D., Melkers, J., Xiao, F. & Pinheiro, D. (2012). A boosted-trees method for name disambiguation. *Scientometrics, 93*(2), 391-411. https://doi.org/10.1007/s11192-012-0681-1

Wasserman, S., & Faust, K. (1994). *Social network analysis*. Cambridge University Press.

Winston, W. L. (1994). *Operations research. Applications and algorithms*. Duxbury Press.

Yang, Y., Wu, M. Z., & Cui, L. (2012). Integration of three visualization methods based on co-word analysis. *Scientometrics, 90*(2), 659-673. https://doi.org/10.1007/s11192-011-0541-4

Zheng, R., Li, J. X., Chen, H. C., & Huang, Z. (2006). A framework for authorship identification of online messages: Writing-style features and classification techniques. *Journal of the American Society for Information Science and Technology, 57*(3), 378-393. https://doi.org/10.1002/asi.20316

APPENDIX

We provide an algorithm in pseudocode to transform $Y = (y_1, \ldots, y_N)$ to $X = (x_1, \ldots, x_N)$ with $Y \preccurlyeq_L X$ by performing basic transfers. We assume that X and Y are ranked in decreasing order and that they are not equal (otherwise, nothing must be done). As X and Y are trees, we know that x_1 and y_1 are both strictly larger than 1 (for $N > 2$) and x_N and y_N are equal to 1.

Algorithm

For $i = 1$ to $N - 1$ (i represents an index):

While $yi < xi$

Find j such that $y_j > x_j$ (otherwise, the transfer cannot lead to the required result)

Apply a basic transfer (transfer by 1) from node y_j to node y_i

Reorder Y

If $Y = X$, the algorithm ends.

Example: For $N = 8$: $Y = (3,3,3,1,1,1,1,1) \preccurlyeq X = (5,3,1,1,1,1,1,1)$.

We take $i = 1$ and observe that $3 < 5$. Next, we see that $y_3 = 3 > x_3 = 1$, hence, $j = 3$. We apply a basic transfer leading to $Y = (4,3,2,1,1,1,1,1)$.

Still with $i = 1$ (as $4 < 5$), we have $j = 3$, with $y_3 = 2 > x_3 = 1$.

Again, we apply a basic transfer leading to $Y = (5,3,1,1,1,1,1,1) = X$.

For the corresponding trees, we have (e.g., Figure A1):

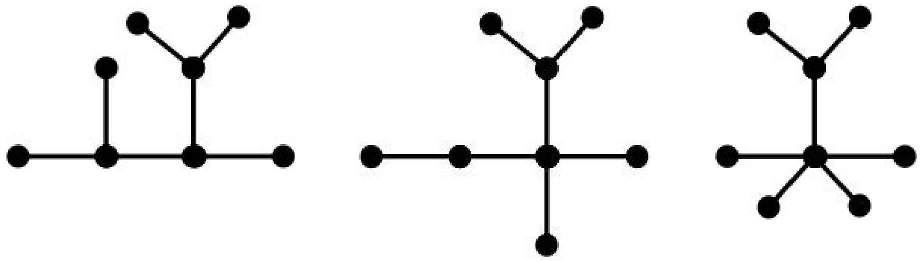

Figure A1. Moving branches to go from Y (left) to X (right).

Next, we provide an example where we start from the eight-node chain leading to $\Delta = (5,2,2,1,1,1,1,1)$, the delta sequence of the first four trees of Figure 3. Recall that the delta sequence of the eight-node chain is (2,2,2,2,2,2,1,1).

Example: Following the algorithm, we see that for $i = 1$, $2 < 5$, hence, $j = 4$ (as $2 > 1$).

This leads to (3,2,2,1,2,2,1,1), and rearranging gives (3,2,2,2,2,1,1,1).

Now, i in the algorithm is still equal to 1 ($3 < 5$) and $j = 4$ ($2 > 1$) leading to (4,2,2,1,2,1,1,1). Rearranging gives (4,2,2,2,1,1,1,1).

Still $i = 1$ ($4 < 5$) and $j = 4$ ($2 > 1$), which leads to $(5,2,2,1,1,1,1,1) = \Delta$.

The corresponding trees are not unique, as they depend on the indexing of the nodes. Figure A2 shows how to get from the chain to a tree with delta sequence Δ. The nodes are indicated by their index number (possibly changing in each step). Observe that for two nodes with equal degrees, we are free to index them as we wish.

Next, we apply a different way of indexing the nodes. This leads to the third tree of Figure 3 (see Figure A3).

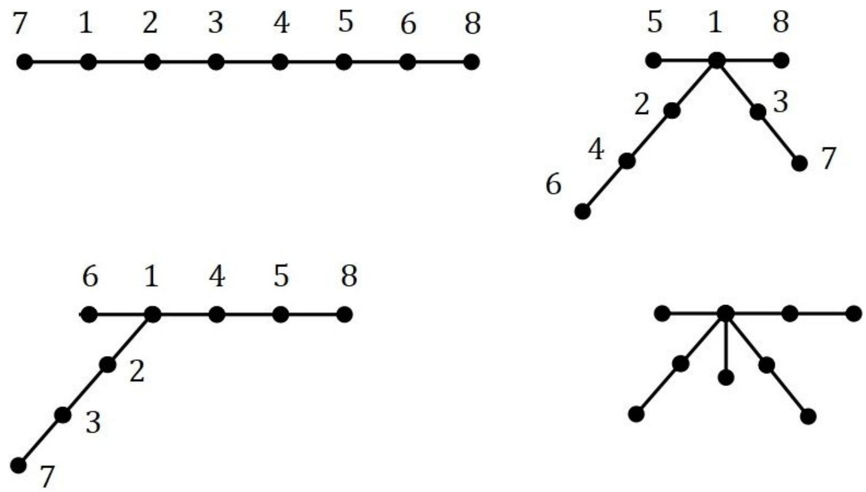

Figure A2. Algorithm leading to the fourth tree in Figure 3.

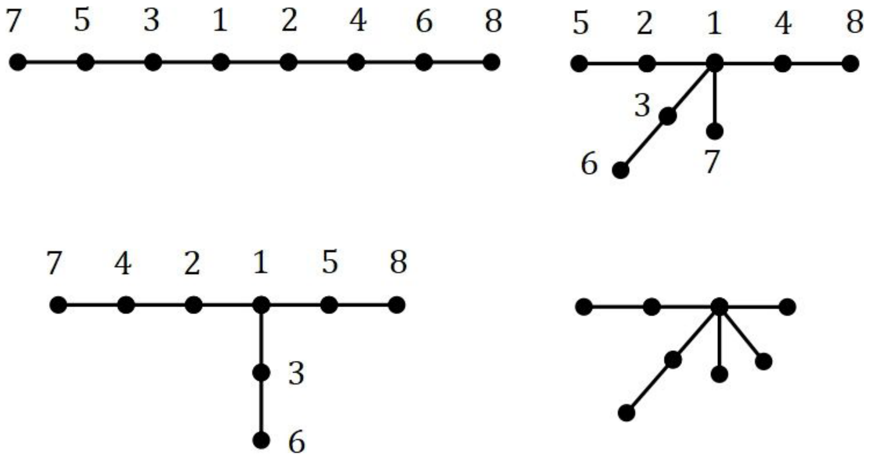

Figure A3. Algorithm leading to the third tree in Figure 3.